

ON STRING GAS COSMOLOGY AT FINITE TEMPERATURE

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Within an adiabatic approximation, thermodynamical equilibrium and a small, nine dimensional, toroidal universe as initial conditions, we analyze the evolution of the dimensions in two different regimes: (i) the Hagedorn regime, with a single scale factor with a nearly constant time evolution (ii) an almost-radiation dominated regime, including the leading corrections due to the lightest Kaluza Klein and winding modes, in which for some initial conditions the large dimensions continue to expand and the small ones remain small.

1 Introduction

In the last decades string theory has become a promised candidate for the underlying theory of the fundamental interactions of nature. However, even though lots of progress has been done, it has not been yet possible to confront it with real physics. One possibility to achieve this is through cosmology by studying the cosmological implications of string theory. On the other hand, string theory provides an alternative to answer the basic questions that standard cosmology currently faces, such as the initial singularity, dimensionality of spacetime, cosmological constant, horizon and flatness problem and the origin of the density perturbations in the cosmic microwave background. All these reasons suggest that cosmology and strings complement each other in several ways, giving rise to string cosmology.

A rigorous way to understand string cosmology is by studying the time dependent backgrounds in string theory,^{1,2,3} unfortunately this is a hard task due to cosmological singularities that arise in this context, and still much work has to be done. On the other hand, a good approximation can be done by studying the adiabatic evolution of time independent backgrounds in string theory. In this direction lots of work have been done where a low energy effective approach is made (see^{4,5} for recent reviews, discussions and references), like for example, Brandenberger and Vafa (B&V) model⁶ in 1988, continuing with the Pre-Big-Bang scenario,⁷ ekpyrotic universes,⁸ brane cosmology,⁹ and so on.

Winding states and *T-duality* are intrinsic features of string theory and play a fundamental role in the B&V model, which main objectives are the solution to the cosmological singularity problem and the dimensionality of

spacetime. In this scenario, the universe starts as a 9 dimensional small torus filled with an ideal gas of strings and evolves such that 6 dimensions remain small and 3 dimensions become large as nowadays. How does it take place? Since we are dealing with a compact universe filled with a string gas, the strings wind around the dimensions. When the system is out of thermal equilibrium there is interaction of this states, in particular, the winding states wrapping around one sense can annihilate with the states wrapping around the other sense, anti-winding states. Now, the key ingredient is that a string describes a 2 dimensional world-sheet, therefore it is easy for the strings to see each other when the dimensionality of space is 3 or less. On the other hand, when the dimensions are more than 3, the strings generically miss each other being unable to annihilate. This, in principle, leads to 3 spatial dimensions free to expand (due to the Kaluza Klein (KK) contribution), and on the other hand, leave the remaining 6 spatial dimensions with winding strings around them preventing them from expansion. On the other hand, since string theory exhibits *T-duality* ($R \rightarrow 1/R$), it implies that neither the temperature nor the physical length are singular as $R \rightarrow 0$, avoiding the cosmological singularity.

Even though this work¹⁰ is inspired by the previous dynamics, we will not study them, but we will be interested in the adiabatic evolution that takes place when the universe is in thermal equilibrium filled with an ideal gas of closed strings (so we are always dealing with weak string coupling). The analysis is performed in such a way that the “string ingredients”, *T-duality* and the presence of winding states, are manifestly present. We study two regimes. First, we address the question of what happens if we start off with 9 dimensional small toroidal universe within a micro-canonical string treatment, Hagedorn regime. It turns out that the dimensions remain nearly constant around the initial value. Second, we assume that by some mechanism, for example, B&V, there are d large and $9 - d$ small dimensions, almost radiation regime, and we study their evolution within a canonical string approach, finding out that for a set of initial conditions it is possible to keep the small dimensions small compared to the large ones (see also¹¹ for a different approach leading to the same conclusion).

2 Set up

In this toy model we consider type **IIA/IIB** closed strings, on a 9 dimensional torus. The starting point is the low-energy effective action,^{12,13,14} with a massless dilaton field which for simplicity is just considered time dependent,

$$\phi = \phi(t),$$

$$S = \int d^{10}x \sqrt{-g} [e^{-2\phi} \{R + 4(\nabla\phi)^2\} + \mathcal{L}_M], \quad (1)$$

where g is the determinant of the background metric $g_{\mu\nu}$, and \mathcal{L}_M corresponds to the matter contribution from the string gas. The metric is taken as $g_{\mu\nu} = \text{diag}(-1, R_1^2(t), \dots, R_9^2(t))$, with scale factors $R_i(t) = e^{\lambda_i(t)}$. It then follows that the dilaton gravity equations of motion¹⁴

$$\begin{aligned} -\sum_{i=1}^9 \dot{\lambda}_i^2 + \dot{\psi}^2 &= e^\psi E, \\ \dot{\lambda}_i - \dot{\psi} \lambda_i &= \frac{1}{2} e^\psi P_i, \\ \ddot{\psi} - \sum_{i=1}^9 \dot{\lambda}_i^2 &= \frac{1}{2} e^\psi E, \end{aligned} \quad (2)$$

where $\psi = 2\phi - \sum_{i=1}^9 \lambda_i$, is the shifted dilaton, and the energy, E , and pressure along the dimension i , P_i , depend on the free energy of the string gas, F , as a function of β (the inverse temperature, $\beta = 1/T$) and the scale factors λ_i ,

$$\begin{aligned} E &= -2 \frac{\delta S_m}{\delta g_{00}} = F + \beta \frac{\partial F}{\partial \beta}, \\ P_i &= -\frac{\delta S_m}{\delta \lambda_i} = -\frac{\partial F}{\partial \lambda_i}. \end{aligned} \quad (3)$$

Finally, since we are interested in an adiabatic evolution, we should impose that the adiabaticity condition $\frac{dS}{dt} = 0$. It is worth pointing out that the equations of motion, eq. (2), and eq. (3) exhibit *T duality* symmetry,^{6,13} $R \rightarrow 1/R$ ($\lambda_i \rightarrow -\lambda_i$, $\phi \rightarrow \phi - \sum_i \lambda_i$), being $R = 1$ the dual radius, since as we said before this gives rise to a non singular cosmology.

The next step in the analysis is the free energy. In analogy to field theory, the free energy of a thermal gas of strings is calculated by computing the one loop world-sheet path integrals for string propagation on $R^{d-1} \times S$, where S corresponds to the euclidean time compactified in a circumference of radius β , which is equivalent to require periodic bosons and anti-periodic fermions.¹⁵ Therefore, the calculation of F is a one loop string calculation of the partition function of type IIA/B string theory compactified on S_1/\mathbf{Z}_2 , where \mathbf{Z}_2 is generated by g , the product of a translation $\pi\beta$ along the circle and $(-1)^{\mathcal{F}}$, with \mathcal{F} the spacetime fermion operator. By means of the unfolding technique,¹⁶ and considering $9-d$ compact dimensions with radius, r_i , the free energy is simplified to

$$F(r_i, \beta) = \frac{V_d}{2\pi} \int_{-1/2}^{1/2} d\tau_1 \int_0^\infty d\tau_2 \frac{1}{\tau_2^{(3+d)/2}} \prod_{i=d+1}^9 \Lambda(r_i, \tau) \sum_{p=1}^\infty |M^2(\tau)| e^{-\frac{\beta^2 p^2}{4\pi\tau_2}}, \quad (4)$$

^aWe have set $\alpha' = 1$.

where τ is the modular parameter of the torus, and the lattice contribution and the spin structure are given by

$$\Lambda(r_i, \tau) = \sum_{m, n=-\infty}^{\infty} q^{\frac{1}{4} \left(\frac{m}{r_i} + nr_i \right)^2} \bar{q}^{\frac{1}{4} \left(\frac{m}{r_i} - nr_i \right)^2}, \quad (5)$$

$$|M^2(\tau)| = \left(\frac{\theta_2^4(\tau)}{\eta^{12}(\tau)} \right)^2 = \sum_{N, \bar{N}=0}^{\infty} D(N) D(\bar{N}) q^N q^{\bar{N}}; \quad (6)$$

where $q = e^{2\pi i \tau}$, $D(N)$ is the degeneracy factor at string level N , m/n are the KK/winding mode along r_i , and θ_2 and η are modular functions on the torus.¹⁷ Eq. (4) diverges for $\beta > \beta_H$, being $\beta_H = 2\sqrt{2}\pi$ the inverse of the Hagedorn temperature.¹⁸ In order to manipulate eq. (4), we should carefully consider the regime we are working in, since when dealing with free string dynamics in the canonical ensemble, fluctuations may diverge as the Hagedorn temperature is approached,¹⁹ invalidating the thermodynamic limit. Therefore in this situation, corresponding to high energy densities, the micro-canonical string ensemble is more physically appropriate. On the other hand, as the energy fluctuations are small compared to the average energy, lower energy densities, we can trustfully work with the canonical ensemble, which is easier to manipulate.

3 Hagedorn regime

In this regime we are dealing with high energy densities and therefore to correctly study the thermodynamics we must make use of the micro-canonical string ensemble. The main quantity to study is the energy density of states $\Omega(E)$,²⁰ which is governed by the analytic structure of the canonical partition function, $Z(\beta) = \text{Tr } e^{-\beta F}$, in the complex β plane,

$$\Omega(E) = \sum_{\alpha} \delta(E - E_{\alpha}) = \int_{L-i\infty}^{L+i\infty} \frac{d\beta}{2\pi i} e^{\beta E} Z(\beta). \quad (7)$$

In this regime we consider all 9 small compact spatial dimensions, with the same radius $r_i = r$ in eq. (4). In order to get an r dependence we need to include, in addition to the leading singularity at β_H , the first next-to-leading singularities at $\beta_W = 2\sqrt{2}\pi(1 - \frac{1}{2r^2})^{1/2}$ and $\beta_K = 2\sqrt{2}\pi(1 - \frac{r^2}{2})^{1/2}$. The partition function, can then be parametrized as

$$Z(\beta) = \frac{e^{\Lambda(\beta, r)}}{\beta - \beta_H} \left(\frac{\eta_K}{\beta - \beta_K} \right)^{18} \left(\frac{\eta_W}{\beta - \beta_W} \right)^{18}, \quad (8)$$

where $\eta_{K/W} = \beta_H - \beta_{K/W}$. Given this, the calculation of the entropy and the thermodynamical quantities is straightforward. For radius close to the dual radius the energy and pressure of the ideal gas of strings take the form

$$\begin{aligned}\frac{1}{T} &\sim \frac{1}{T_H} + C_1 E^{17} e^{-\eta E}, \\ P &\sim C_2 E^{17} e^{-\eta E},\end{aligned}\tag{9}$$

where C_1 and C_2 are polynomial functions of the singularities, $\eta \approx \sqrt{2}\pi[2 - (4 - \frac{2}{r^2})^{1/2}]$ and $T_H = 1/\beta_H$. As we see, temperature and pressure exhibit an exponential suppression coming from the energy and therefore at high energy densities the pressure is negligible and the temperature is nearly constant corresponding to the Hagedorn one. The string gas then, in this regime, presents an equation of state corresponding to that of pressureless dust and then, the time evolution of the dimensions is such that they remain nearly constant close to the dual radius. In other words, in order that the dimensions present interesting dynamics, such as expansion or contraction, the system must get out of thermal equilibrium in accordance with B&V model.

We can impose as well the conservation of the total winding number n_i and the discrete momenta m_i along each compact spatial dimension by introducing a chemical potential μ for each conserved charge Q ,²¹ such that $Z(\beta, \mu) = \text{Tr} e^{-\beta F + 2\pi i \mu Q}$, with $\mu Q \rightarrow \mu_i m_i + \nu_i n_i$. In contrast with the case without conservation, already the leading singularity depends on the compactification radius, $2\beta_H(\mu_i, \nu_i, r_i) = \sqrt{\beta_0^2 + \sum_i (\frac{\mu_i}{r_i} + \nu_i r_i)^2} + \sqrt{\beta_0^2 + \sum_i (\frac{\mu_i}{r_i} - \nu_i r_i)^2}$, where $\beta_0 = 2\pi\sqrt{2}$. However, for high energies the pressureless dust behavior is recovered.

4 Almost radiation regime

In the previous section, once the system gets out of thermal equilibrium it is possible that interesting dynamics arise. For example, annihilation of winding and anti-winding states would take place in some dimensions letting them free to expand, whereas the other dimensions would be unable to expand due to the windings around them. If we suppose such a mechanism took place, we could ask how the dynamics are once the system gets again into thermal equilibrium. More explicitly, if we start off with some large and some small dimensions, how is their evolution in thermal equilibrium? Is it possible to keep this hierarchy on sizes?

At this point, since we are dealing with lower temperatures, we are able to use the canonical string ensemble, which deals with the canonical parti-

tion function. The realization of d large and $9 - d$ small dimensions in the calculation, is given by the fact that only the small compact dimensions feel the contribution of winding and KK modes in the partition function. The free energy, eq. (4), can be split in *a*) the contribution from the zero modes ($N = \bar{N} = m_i = n_i = 0$) which we refer to as *radiation* contribution, *b*) *matter* contribution coming from the non-zero modes ($N, \bar{N}, m_i, n_i \neq 0$).

Concerning the radiation contribution it turns out that only the large dimensions do feel it. More explicitly, the large dimensions present a radiation-like equation of state, $P_d^{\text{rad}} = E^{\text{rad}}/d$, whereas the pressure for the small dimensions is zero, $P_{9-d}^{\text{rad}} = 0$. This translates into the fact that the large dimensions are able to expand, independent of the initial conditions^b, whereas the small dimensions stay around their initial value.

On the other hand, the massive contribution affects both, large and small dimensions. For simplicity, we just considered the lightest KK and winding modes along the compact dimensions. The energy and pressures are given in terms of modified Bessel functions, which are functions of the KK and winding modes,

$$E^{\text{mat}} \sim \frac{f(\beta)}{r^{(d+1)/2}} [K_{(d+1)/2}(\beta/r) + \beta/r K'_{(d+1)/2}(\beta/r) + r^{d+1} \{K_{(d+1)/2}(\beta r) + \beta r K'_{(d+1)/2}(\beta r)\}], \quad (10)$$

$$P_d^{\text{mat}} = \frac{f(\beta)}{r^{(d+1)/2}} [K_{(d+1)/2}(\beta/r) + r^{d+1} K_{(d+1)/2}(\beta r)], \quad (11)$$

$$P_{9-d}^{\text{mat}} = \frac{g(\beta)}{r^{(d+3)/2}} [K_{(d-1)/2}(\beta/r) - r^{d+3} K_{(d-1)/2}(\beta r)], \quad (12)$$

where $f(\beta)$ and $g(\beta)$ are functions of β and the prime denotes derivative with respect to r . As seen in eq. (11) and eq. (12), the winding modes lead to a positive contribution to the large dimensions and to a negative to the small ones, contrary to the KK modes which give positive contribution to all dimensions.

Whereas eq. (11) always leads to an expansion of the large dimensions, the effect of eq. (12) depends on the initial size of the small dimensions. For initial radius larger than the dual radius, (a) in fig. 4, the pressure, eq. (12), is such that it makes the small dimensions expand. In other words, the probability of having winding modes along the compact dimensions is smaller as the radius is larger, letting the dimensions free to expand. On the other hand, for initial radius less than the dual one, the probability of having winding modes

^bEven for negative initial expansion rate, the large dimensions initially contract, bounce and finally expand.

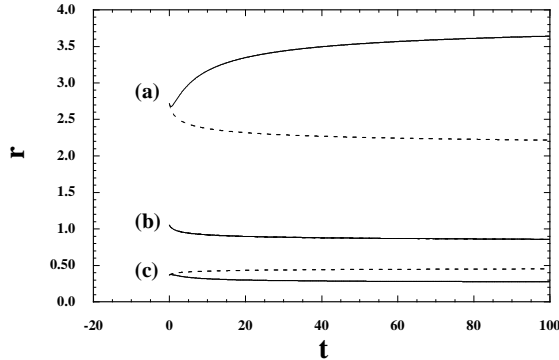


Figure 1. Evolution of the small dimensions in the almost radiation regime. a) $\nu_0 = 1$, $\dot{\nu}_0 = -0.1$, b) $\nu_0 = 0.05$, $\dot{\nu}_0 = -0.1$, c) $\nu_0 = -1.0$, $\dot{\nu}_0 = 0.1$, $\psi_0 = -16$, $\beta_0 = 12$.

is larger, leading to the contraction of the small dimensions, (b) in fig. 4. Finally, for radius close to the dual one^c, (c) in fig. 4, the effect of the winding modes is compensated by that of the KK modes resulting in zero pressure, and giving rise to the stabilization of the small dimensions. Fig. (4), shows the behavior for the different choice of initial conditions with the contribution of pure radiation (dotted lines) and the contribution of pure radiation and matter (solid lines).

We therefore conclude that in the almost radiation regime, as long as the small dimensions start close to the self dual radius, they remain small whereas the large dimensions expand. It is good to point out that these results are independent of the number of large/small dimensions. Finally, as in the Hagedorn case, imposing the conservation of KK and winding modes does not change the picture quantitatively.

Acknowledgments

I would like to thank S. Tsujikawa, B. Bassett and specially M. Serone for their collaboration in the work this talk is based on. I am also happy to thank M. Blau for useful discussions.

^cThinking about B&V model, this is the case we are interested in, since the small dimensions are supposed to be around the self dual radius.

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